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Analysis of Economic Growth Models via Value Function Design ⋆

Alexander L. Bagno * **Alexander M. Tarasyev** **

* *Ural Federal University, Yekaterinburg, Russia (e-mail: bagno.alexander@gmail.com)*

** *Krasovskii Institute of Mathematics and Mechanics of UrB RAS, Ural Federal University, Yekaterinburg, Russia (e-mail: tam@imm.uran.ru)*

Abstract: Properties of the value function are examined in an infinite horizon optimal control problem with an unlimited integrand index appearing in the quality functional with a discount factor. Optimal control problems of such type describe solutions in models of economic growth. Necessary and sufficient conditions are derived to ensure that the value function satisfies the infinitesimal stability properties. It is proved that value function coincides with the minimax solution of the Hamilton–Jacobi equation. Description of the growth asymptotic behavior for the value function is provided for the logarithmic, power and exponential quality functionals. An example is given to illustrate construction of the value function in economic growth models.

Keywords: Optimal control, value function, stability properties, Hamilton–Jacobi equations.

1. INTRODUCTION

Stability conditions play a key role in the theory of optimal control and differential games. They allow to find the optimal control on the feedback principle. They also serve as a basis for developing finite difference operators in grid methods of the value function approximation (Ushakov, 1994).

Stability conditions have been introduced by N. N. Krasovskii and A. I. Subbotin in the differential games theory (Krasovskii, 1974) and generalized in the infinitesimal form in the theory of generalized nonsmooth solutions (minimax solutions, viscosity solutions) of partial differential equations of the Hamilton–Jacobi type (Crandall (1983), Subbotin (1991), Subbotina (2013)).

The article is devoted to the study of connection between the value function in optimal control problems with infinite horizon (Aseev, 2007) and infinitesimal stability properties. Optimal control problems with infinite horizon arise as a basic element of analysis in economic growth models and in problems of motion stabilization. Particularly, we examine the optimal control problem with a nonlinear dynamics and with a discounted integral quality functional which is presented by an unbounded function.

Let us note that similar problems were considered in the papers, by C. Dolcetta (1984), by R. Adiatullina (1987), by Subbotin (1985), by A. Tarasyev (1999) and by G. Klaassen (2004). This article is a development of the papers A. Bagno, A. Tarasyev (2016) and Nikolskii (2004), where properties of the asymptotic growth are examined for the value function in the case of unbounded quality indices. In these papers, the continuity property is proved for the value function in infinite horizon problems and some estimates are obtained for Hölder parameters of continuity.

2. SYSTEM DYNAMICS AND VALUE FUNCTION

The article deals with the following dynamic control system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = 0, \tag{1}$$

where $x \in R^n$, $u \in P \subset R^p$ (P is a compact set). The quality functional is given by the relation

$$J(x(\cdot), u(\cdot)) = \int_{t_0}^{+\infty} e^{-\lambda\tau} h(x(\tau), u(\tau)) d\tau, \quad \lambda > 0,\ t_0 > 0. \tag{2}$$

It is assumed that the following conditions are satisfied.

(1) Functions f and h are continuous in the set of variables on $R^n \times P$.
(2) The Lipschitz condition in argument x is valid for all $x_1, x_2 \in R^n$, and for all p.
$$\|f(x_1, p) - f(x_2, p)\| \leq L\|x_1 - x_2\|,$$
$$|h(x_1, p) - h(x_2, p)| \leq L\|x_1 - x_2\|, \tag{3}$$
where L is a Lipschitz constant.
(3) The condition of the sublinear growth in argument x is valid for all x, p:
$$\|f(x, p)\| \leq \kappa(1 + \|x\|), \tag{4}$$
$$|h(x, p)| \leq \kappa(1 + \|x\|), \tag{5}$$
where κ is a positive constant.

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For convenience let us introduce the parameter
$$\widetilde{y} = e^{-\lambda\tau} h(x(\tau), u(\tau)).$$

We define the value function according to C. Dolcetta (1984). Let $u(\cdot)$ be a Lebesgue measurable program control on a finite interval $[t_0, T]$. Denote by U_T the set of controls $u(\cdot)$ on the interval $[t_0, T]$.

The Value function of finite horizon problem for the initial point (t_0, z_0), when $t_0 \in (0, T)$, $z_0 = \begin{pmatrix} x_0 \\ \widetilde{y}_0 \end{pmatrix}$, $x_0 \in R^n$, $\widetilde{y}_0 \in R$, is defined by the relation

$$w_T(t_0, z_0) = \sup_{u(\cdot)\in U} \left(\widetilde{y}_0 + \int\limits_{t_0}^{T} e^{-\lambda\tau} h(x(\tau), u(\tau))\, d\tau \right),$$

when $x(\cdot)$ satisfies the dynamics (1) on the interval $[t_0, T]$, $x(t_0) = x_0$.

In addition, we determine the value function of the infinite horizon problem. Denote by U the set of all Lebesgue measurable program controls $u(\cdot)$ on infinite interval $[t_0, +\infty)$.

The Value function of infinite horizon problem for the starting point (t_0, z_0), when $t_0 \in (0, T)$, $z_0 = \begin{pmatrix} x_0 \\ \widetilde{y}_0 \end{pmatrix}$, $x_0 \in R^n$, $\widetilde{y}_0 \in R$, is defined by the relation

$$w(t_0, z_0) = \\ = \sup_{u(\cdot)\in U} \lim_{T\to+\infty} \left(\widetilde{y}_0 + \int\limits_{t_0}^{T} e^{-\lambda\tau} h(x(\tau), u(\tau))\, d\tau \right), \quad (6)$$

when $x(\cdot)$ satisfies the dynamics (1) on the interval $[t_0, +\infty])$, $x(t_0) = x_0$.

Let us note that

$$w_T(t_0, z_0) = \\ = -\inf_{u(\cdot)\in U} \left(-\widetilde{y}_0 - \int\limits_{t_0}^{T} e^{-\lambda\tau} h(x(\tau), u(\tau))\, d\tau \right) = \\ = -\widetilde{y}_0 - \inf_{u(\cdot)\in U} \left(\int\limits_{t_0}^{T} e^{-\lambda\tau} (-h(x(\tau), u(\tau)))\, d\tau \right).$$

If we define the function $g(x, u) = -h(x, u)$, $y = -\widetilde{y}$, $x \in R^n$, $u \in P$, then we can write

$$w_T(t_0, z_0) = \\ = -\widetilde{y}_0 - \inf_{u(\cdot)\in U} \lim_{T\to+\infty} \left(\int\limits_{t_0}^{T} e^{-\lambda\tau} g(x(\tau), u(\tau))\, d\tau \right) = \\ = -\omega_T(t_0, z_0)$$

and consider the function $\omega_T(t_0, z_0)$ as the value function. Function h satisfies properties (3) and (5), and the following relation is valid
$$y = -\widetilde{y} = -e^{-\lambda\tau} h(x(\tau), u(\tau)) = e^{-\lambda\tau} g(x(\tau), u(\tau))$$
for parameter y.

We introduce for function $w(t_0, z_0)$ its analogue

$$w(t_0, z_0) = \\ = -\inf_{u(\cdot)\in U} \lim_{T\to+\infty} \left(-\widetilde{y}_0 - \int\limits_{t_0}^{T} e^{-\lambda\tau} h(x(\tau), u(\tau))\, d\tau \right) = \\ = -y_0 - \inf_{u(\cdot)\in U} \lim_{T\to+\infty} \left(\int\limits_{t_0}^{T} e^{-\lambda\tau} (-h(x(\tau), u(\tau)))\, d\tau \right) = \\ = -\omega(t_0, z_0)$$

Our task is to analyze properties of the value function $\omega(t_0, z_0)$.

3. STABILITY PROPERTIES OF VALUE FUNCTION

Let us introduce some notations and definitions. Let $t \in [0, +\infty)$, $u \in P$, $z = (x, y) \in R^m \times R$, $S = \{s \in R^m : \|s\| = 1\}$. Assume
$$F_1(t, x) = \mathrm{co}\{(f(x, u), e^{-\lambda t} g(x, u)) : u \in P\},$$
$$F_2(t, x, u) = (f(x, u), e^{-\lambda t} g(x, u)).$$
Here, the symbol $\mathrm{co}\{x : x \in X\}$ denotes the convex hull of X.

Let us define the Hamiltonian of the control problem by the relation
$$H(x, s) = \frac{1}{\lambda} \min_{u\in P} (\langle s, f(x, u)\rangle + g(x, u)).$$

We denote by the symbol $Z_1(t, z)$ the set of absolutely continuous functions $z(\cdot)$, mapping the interval $[t, +\infty)$ to R^{m+1} and satisfying almost everywhere the differential inclusion
$$\dot{z}(\tau) \in F_1(\tau, x(\tau)),$$
with the initial condition $z(t) = z_0$, when $\tau \in [t, +\infty]$.

We denote by the symbol $Z_2(t, z, u)$ the set of absolutely continuous functions $z(\cdot)$, mapping the interval $[t, +\infty)$ to R^{m+1} and satisfying almost everywhere the differential inclusion
$$\dot{z}(\tau) = F_2(\tau, x(\tau), u), \quad u \in P,$$
with the initial condition $z(t) = z_0$, when $\tau \in [t, +\infty]$.

We follow the terminology (Krasovskii, 1974), when motions in the sets $Z_1(t, z)$, $Z_2(t, z, u)$ are called *generalized motions*.

The generalized motions satisfy the following statement.

Lemma 1. Let $t_0 \in [0, +\infty)$, $z_0 = (x_0, y_0) \in R^n \times R$, and motion $z(t) = (x(t), y(t))$, where $y(t) = e^{-\lambda t} g(x(t), u(t))$, is located in the set $Z_1(0, (x_0, 0))$, then motion $z_*(t) = (x(t - t_0), e^{-\lambda t_0} y(t - t_0) + y_0)$ also lies in the set $Z_1(0, (x_0, 0))$, here $t \in [t_0, +\infty)$.

Proof. Proof. By the definition of the set $Z_1(0, (x_0, 0))$, we have

$$\dot{z}(\tau - t_0) \in \\ \mathrm{co}\{f(x(\tau - t_0), u), e^{-\lambda(\tau - t_0)} g(x(\tau - t_0), u)) : u \in P\}$$
for almost all $\tau > t_0$. The function $z_*(t)$ is defined as
$$z_*(t) = (x(t - t_0), e^{-\lambda t_0} y(t - t_0) + y_0),$$
therefore, it satisfy the relation
$$\dot{z_*}(\tau) \in \mathrm{co}\{f(x_*(\tau), u), e^{-\lambda(\tau)} g(x_*(\tau), u)) : u \in P\}.$$

Since
$$z_*(t_0) = (x(0), e^{-\lambda t_0} y(0) + y_0) = (x_0, y_0),$$
then motion $z_*(\cdot)$ lies in the set $Z_1(0,(x_0,0))$. Lemma is proved.

Theorem 2. Let $\lambda > \kappa$, then the following estimate
$$|\omega(t_0, z_0)| \le A + B\|x_0\| \tag{7}$$
is true for the value function of the optimal control problem with infinite horizon. Here $A = |y_0| + \frac{\kappa}{\lambda} e^{-\lambda t_0}$, $B = \frac{1}{\lambda-\kappa} e^{-\lambda t_0}$, $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Proof. Let $\omega(t_0, z_0)$ be the value function in the problem with infinite horizon. Let us show the estimate (7).

According to definition of the value function (6)
$$|\omega(t_0, z_0)| \le \sup_{u \in U} \lim_{T \to \infty} \left| y_0 + \int_{t_0}^{T} e^{-\lambda\tau} g(x(\tau), u(\tau))\, d\tau \right| \le$$
$$\le |y_0| + \sup_{u \in U} \lim_{T \to \infty} \int_{t_0}^{T} |e^{-\lambda\tau} g(x(\tau), u(\tau))|\, d\tau \le$$
$$\le |y_0| + \sup_{u \in U} \lim_{T \to \infty} \int_{t_0}^{T} \kappa e^{-\lambda\tau} (1 + \|x(\tau)\|)\, d\tau \le$$
$$\le |y_0| + \lim_{T \to \infty} \int_{t_0}^{T} \kappa e^{-\lambda\tau} (1 + \|x_0\| e^{\kappa(\tau - t_0)})\, d\tau =$$
$$= |y_0| + \int_{t_0}^{+\infty} \kappa e^{-\lambda\tau}\, d\tau + \int_{t_0}^{+\infty} \kappa \|x_0\| e^{-\lambda\tau + \kappa(\tau - t_0)}\, d\tau =$$
$$= |y_0| + \frac{\kappa}{-\lambda} e^{-\lambda\tau} \Big|_{t_0}^{+\infty} + \frac{\|x_0\|}{-\lambda + \kappa} e^{-\lambda\tau + \kappa(\tau - t_0)} \Big|_{t_0}^{+\infty} =$$
$$= |y_0| + \frac{\kappa}{\lambda} e^{-\lambda t_0} + \frac{\|x_0\|}{\lambda - \kappa} e^{-\lambda t_0} =$$
$$= A + B\|x_0\|.$$
Thus, we get the estimate (7). Let us note that the condition $\lambda > \kappa$ provides the convergence of the indicated integrals. Theorem is proved.

Let us consider special cases, when the quality functional includes the index $-h(x,u) = g(x,u)$ of one of the following three types. The control parameter solves the maximization problem in these cases: $G(x) = \max_{u \in P} |g(x,u)|$. Let us note that these cases arise in economic growth models (see (, Intriligator, 2002, p. 167)).

1. $G(x) = \max\{0, -ke^{-\sum_{i=1}^{n} a_i x_i} + k\}$, where $k > 0$, $a_i > 0$, $t \in [0,T]$. The value function has the following estimate in this case
$$|\omega(t_0, z_0)| \le 0 - ke^{-\|x_0\|} + \frac{k}{\lambda} e^{-\lambda t_0}.$$

2. $G(x) = \sum_{i=1}^{n} \frac{a_i}{1-b_i} x_i^{1-b_i}$, where $0 < b_i < 1$, $a_i > 0$, $t \in [0,T]$. The value function is limited from above by the estimate
$$|\omega(t_0, z_0)| \le \frac{\beta \|x_0\|^{\alpha} e^{-\alpha\kappa t_0}}{\alpha\kappa - \lambda} (-e^{(\alpha\kappa - \lambda)t_0}).$$

3. $G(x) = \max\{0, \sum_{i=1}^{n} a_i \ln x_i\}$, where $a_i > 0$, $t \in [0,T]$. In this case, we obtain the estimate
$$|\omega(t_0, z_0)| \le \frac{e^{-\lambda t_0}}{\lambda} \max\{0, \ln \|x_0\|\} + \kappa \sum_{i=1}^{n} a_i \frac{e^{-\lambda t_0}}{\lambda^2}.$$

Let us note that the condition $\lambda > \kappa$ is not used in all three cases.

It is proved in (A. Bagno, A. Tarasyev, 2016, lemma 4) that the value function $\omega(t,z)$ can be presented as
$$\omega(t,z) = y + e^{-\lambda t} \omega(0, x, 0).$$
Let introduce the function $v(x) = \omega(0,(x,0))$ and call it *stationary value function.*

Theorem 3. Let $\lambda > \kappa$, then function $\varphi : R^m \to R$ is the value function of the optimal control problem (1) (2), if and only if the following conditions are valid:

(1) the function φ is continuous and bounded
$$|\varphi(x)| \le A + B\|x\|.$$
(2) motions $z(t) = (x(t), y(t))$ from the set $Z_1(0,(x,0))$, where $y(t) = e^{-\lambda t} g(x(t), u(t))$, satisfy the inequality
$$e^{-\lambda t} \varphi(x(t)) + y(t) \le \varphi(x) \tag{8}$$
for all $t \in [0, +\infty)$, $x \in R^n$.
(3) motions $z(t) = (x(t), y(t))$, $y(t) = e^{-\lambda t} g(x(t), u(t))$, from the set $Z_2(0,(x,0),u)$, satisfy the inequality
$$e^{-\lambda t} \varphi(x(t)) + y(t) \ge \varphi(x) \tag{9}$$
for all $t \in [0, +\infty)$, $x \in R^n$.

Proof. Let φ be the value function of the problem (1) (2). Let us show that conditions 1–3 are true. We start with the first condition and have the relation
$$|\varphi(x_0)| = |\omega(t_0, x_0)| \le A + B\|x_0\|$$
according to the theorem 2.

Let us pass to condition 2 and prove it. According to definition of the value function, there exists strategy u such that
$$\lim_{\theta \to +\infty} y(\theta) \le w(x_0) + \varepsilon$$
for all positive ε and for all $z(\cdot) = (x(\cdot), y(\cdot))$. Therefore,
$$y_0(t) + e^{-\lambda t} w(x_0(t)) = \omega(t, z_0(t))$$
$$= \sup_{u} \lim_{\theta \to +\infty} y(\theta) \le w(x_0) + \varepsilon,$$
here $u \in U$, $z(\cdot) = (x(\cdot), y(\cdot))$. Thus, there exists motion z_0 which satisfies condition 2 for all positive ε.

Proof of condition 3 is similar.

Now, we prove the converse statement. Let function φ satisfy conditions 1–3, and $x_0 \in R^m$. Let us show that $\varphi(x_0) = \omega(x_0)$. There exists motion $z(\cdot) = (x(\cdot), y(\cdot)) \in Z_1(0,(x_0,0))$ such that
$$e^{-\lambda(t-t_0)} \varphi(x(t-t_0)) + y(t-t_0) \le \varphi(x_0).$$
by condition 2. Double using condition 2, we get
$$e^{-\lambda t} \varphi(x(t-t_0)) + e^{-\lambda t_0} y(t-t_0) + y_0$$

$$\leq e^{-\lambda t_0}\varphi(x_0) + y_0 \leq \varphi(x_0).$$

Motion

$$(x(\tau - t_0), e^{-\lambda t_0} y(\tau - t_0) + y_0)$$

is located in the set $Z_1(t_0, x_0)$ by lemma 1, when $\tau \in [t_*, +\infty)$, $z_0 = (x_0, y_0)$. Now we can use lemma 15.1 (Krasovskii, 1974). Namely, there exists strategy u such that inequality

$$e^{-\lambda T}\varphi(x(T)) + y(T) \leq \varphi(x_0)$$

is valid for all motions $z(\cdot) = (x(\cdot), y(\cdot))$ and for all $T \in (0, +\infty)$. Using theorem 2, we can write

$$-e^{-\lambda T}(A + B\|x(T)\|) + y(T) \leq \varphi(x_0).$$

Then we get

$$-e^{-\lambda T}(A + B\|x(T)\|) + \omega_T(0, (x_0, 0)) \leq \varphi(x_0)$$

by the definition of the value function $\omega_T(0, (x_0, 0))$, when $T \in (0, +\infty)$. Using the Grönwall inequality, we obtain

$$-e^{-\lambda T}A - e^{-\lambda T}B\|x_0\|e^{\kappa(\tau - t_0)} + \omega_T(0, (x_0, 0)) \leq \varphi(x_0),$$

when $T \in (0, +\infty)$. We pass to the limit by tending T to infinity. The first term obviously tends to zero. The second term also tends to zero, since $\lambda > \kappa$. As a result, we obtain

$$\omega(0, (x_0, 0)) \leq \varphi(x_0).$$

Similarly, we can show that

$$\omega(0, (x_0, 0)) \geq \varphi(x_0).$$

Theorem is proved.

4. VALUE FUNCTION AS MINIMAX SOLUTION OF HAMILTON–JACOBI EQUATION

In this section, we consider the Hamilton–Jacobi equation

$$-\varphi + \frac{1}{\lambda}\min_u(\langle \nabla\varphi, f(x,u)\rangle + g(x,u)) = 0. \qquad (10)$$

Let us introduce some notations and definitions

$$S = \{s \in R^n : \|s\| = 1\},$$
$$A(x) = \{f \in R^n : \|f\| \leq \sqrt{2}\kappa(1 + \|x\|)\},$$
$$A_u(x,q) = \{f \in A(x) : \langle f, q\rangle \geq H(x,q)\}$$
$$A_l(x,p) = \{f \in A(x) : \langle f, p\rangle \leq H(x,p)\},$$

where $q, p \in R^n$. We assume that the condition $\lambda > \kappa$ is true. Let us denote by the symbol $X_u(x,q)$ $(X_l(x,p)$ a set of absolutely continuous functions satisfying differential inclusions for almost all t

$$\dot{x}(t) \in A_u(x,q) \quad (\dot{x}(t) \in A_l(x,p)).$$

Upper solution of the equation (10) is defined as a lower semicontinuous function $\varphi : R \times R^n \to R$, for which there exists function $z_u = (x_u(\cdot), y_u(\cdot)) \in X_u(x,q)$ satisfying condition (11) for all $x \in R^n$, $\tau > 0$, $q \in S$

$$e^{-\tau}\varphi(x_u(\tau)) + y_u(\tau) \leq \varphi(x). \qquad (11)$$

Lower solution of the equation (10) is defined as an upper semicontinuous function $\varphi : R \times R^n \to R$, for which there exists function $z_l = (x_l(\cdot), y_l(\cdot)) \in X_l(x,p)$ satisfying condition (12) for all $x \in R^n$, $\tau > 0$, $p \in S$

$$e^{-\tau}\varphi(x_l(\tau)) + y_l(\tau) \geq \varphi(x). \qquad (12)$$

A continuous function $\varphi : R \times R^n \to R$, which is an upper and lower solution of the equation (10) simultaneously, is called *minimax solution* of the equation.

The lower (upper) Dini derivative in a direction d is defined by the following relations

$$\partial_-\omega(x)|(d) = \lim_{\varepsilon\to 0}\inf_{(\delta,d')\in\Delta_\varepsilon(x,d)}\frac{\omega(x+\delta d') - \omega(x)}{\delta}$$

$$\left(\partial_+\omega(x)|(d) = \lim_{\varepsilon\to 0}\sup_{(\delta,d')\in\Delta_\varepsilon(x,d)}\frac{\omega(x+\delta d') - \omega(x)}{\delta}\right),$$

where $\Delta_\varepsilon(x,d) = \{(\delta, d') \in (0,\varepsilon) \times R^n : \|d - d'\| \leq \varepsilon\}$.

The following two statements are valid for the stability properties (8), (9) Sultanova (1995).

Claim 4. Let φ be a continuous function. Then conditions 8 and 9 are equivalent to conditions 13, 14:

$$\min_{d=(d_1,d_2)\in A_u(x,q)}\{d_2 + \partial_-\varphi(x)|(d_1)\} - \varphi(x) \leq 0, \quad (13)$$

$$\max_{d=(d_1,d_2)\in A_l(x,p)}\{d_2 + \partial_+\varphi(x)|(d_1)\} - \varphi(x) \geq 0. \quad (14)$$

Claim 5. Let φ be a continuous function. Then conditions 13, 14 are equivalent to conditions 15, 16:

$$\sup_{d\in R}\{\langle s, d\rangle - \partial_-\varphi(x)|(d)\} \geq -\varphi(x) + H(x,s), \quad (15)$$

$$\inf_{d\in R}\{\langle s, d\rangle - \partial_+\varphi(x)|(d)\} \leq -\varphi(x) + H(x,s). \quad (16)$$

Lemma 6. Any lower solution of the equation 10, satisfying sublinear growth conditions 4, 5, does not exceed any upper solution, that is $\forall x \in R^n$ the following relation holds

$$\varphi_u(x) \geq \varphi_l(x).$$

Theorem 7. Let a minimax solution of equation 10 satisfy the sublinear growth condition 7. Then this solution is unique.

This theorem is proved analogously to results presented in Subbotin (1991).

Let us remind that the function $v(x) = \omega(0, (x, 0))$ is called the stationary value function. Using results of the theory of minimax solutions Subbotin (1991) one can prove the following statement.

Theorem 8. The value function v of the problem 1, 2 is the minimax solution of the equation 10

$$-\varphi + \frac{1}{\lambda}\min_u(\langle \nabla\varphi, f(x,u)\rangle + g(x,u)) = 0,$$

that satisfies the sublinear growth condition 7.

4.1 EXAMPLE OF VALUE FUNCTION DESIGN

Let us consider the nonlinear dynamic system

$$\frac{\dot{x}(\tau)}{x(\tau)} = f(\tau) - g(\tau)\,\frac{u(\tau)}{x(\tau)}, \quad x(t_0) = x_0. \qquad (17)$$

This model involves production $x = x(\tau)$, production rate $\frac{\dot{x}}{x}$, technology rate $u = u(\tau)$ and functions $f(\tau)$ and $g(\tau)$ that depends of production factors: labor, capital, materials and energy. The function $g(\tau) = p(\tau) - q(\tau)$ describes the difference between decrease in manufacturing due to R&D spending $p(\tau)$ and increase of R&D knowledge stock $q(\tau)$. We consider equation (17) as a balance equation of spending resources between the productivity rate $\frac{\dot{x}}{x}$ and R&D intensity $\frac{\dot{u}}{x}$. The negative sign $(-g(\tau))$ of the net contribution by R&D means that, in the short-run, the technology consumption $p(\tau)$ exceeds the rate $q(\tau)$ of return to R&D. The technology rate u is a control parameter.

Now we need to introduce the utility function represented by the integral with the discount coefficient λ

$$U = \int_{t_0}^{+\infty} e^{-\lambda(\tau - t_0)} \ln D(\tau)\, d\tau. \tag{18}$$

Here $\ln D(\tau) = (\ln x(\tau) + A \ln u(\tau))$, $A = (1-\alpha)/\alpha$. The consumption index $D(\tau)$ represents an utility of technologies at time τ, the variable α stands for the parameter of elasticity $(0 < \alpha < 1)$, t_0 is the initial time and τ is the running time. The structure of the utility function (18) means that investors are interested both in growth of production x and in growth of new products which is provided by the technology rate u. The balance equation (17) describes the dynamical relation between the production x and the technology rate u, and gives restrictions on growth of the technology rate u.

The problem is to find such technology rate $u^0(\tau)$ in the class of piecewise-constant functions $u(\tau)$ and the corresponding optimal production $x^0(\tau)$ subject to dynamics (17), that maximize the utility function (18).

Applying the Pontryagin maximum principle to the control problem (17), (18) and assuming that function $f(\tau)$ is a nondecreasing with the positive value of the difference $(1-\alpha)\lambda$ we get the optimal solution with the exponentially growing production x

$$x(\tau) = x_0 e^{Q(\tau)}, \quad x_0 = x(t_0),$$

where

$$Q(\tau) = \int_{t_0}^{\tau} (f(s) - (1-\alpha)\lambda) ds.$$

We obtain also relations between the optimal investment $u = u^0$ and the optimal production $x = x^0$

$$u(\tau) = \frac{(1-\alpha)\lambda}{g(\tau)} x(\tau).$$

This equation means that the optimal R&D investment u increases proportionally to growth of the optimal production x with the coefficient of proportionality $(1-\alpha)\lambda/g(\tau)$. For R&D intensity u/x we have the following formula

$$\frac{u(\tau)}{x(\tau)} = \frac{(1-\alpha)\lambda}{g(\tau)},$$

which describes the dependence of the optimal R&D intensity on the substitution parameter α, the subjective discount rate λ and the discounted marginal productivity of technology $(-g(\tau))$. Let us note that, when the cost $p(\tau)$ for sustaining the accumulated R&D investment is high, then the research intensity u/x is low. Vice versa, increase of the rate of return to R&D $q(\tau)$ leads to the growth of the research intensity u/x. Assuming that the positive function $g(\tau)$ is nonincreasing over time τ we get the growth property of the research intensity u/x.

Finally, we consider the value function $(t, y) \to \varphi(t, y)$, which assigns the optimal result φ of the utility function (18) along the optimal process $(x^0(\tau), u^0(\tau))$ of the dynamics (17) to an initial position (t, y), $t = t_0$, $y = x_0$.

Let us calculate the value function in our example as the minimax solution of the Hamilton–Jacobi equation

$$\frac{\partial \varphi(t,y)}{\partial t} + \frac{\partial \varphi(t,y)}{\partial y} f(t) y + e^{-\lambda t} \ln y +$$
$$+ \max_u \Big\{ -\frac{\partial \varphi(t,y)}{\partial y} g(t) u + e^{-\lambda t} A \ln u \Big\} = 0. \tag{19}$$

Using the method of indeterminate coefficients for the partial differential equation of the Hamilton–Jacobi type (19) we obtain the analytical formula for the minimax solution which coincides with the value function

$$\varphi(t, y) = e^{-\lambda t} \big(\mu(y) + \nu(t)\big), \tag{20}$$

$$\mu(y) = \frac{(A+1)}{\lambda} \ln y, \quad \nu(t) = -\int_t^{+\infty} e^{-\lambda(\tau - t)} h(\tau) d\tau. \tag{21}$$

Here

$$h(\tau) =$$
$$= A \ln g(\tau) - \frac{A+1}{\lambda} f(\tau) - A(\ln(1-\alpha) - \ln \lambda - 1) \tag{22}$$

In particular, if h is a constant, $h(\tau) \equiv h$, then ν is also a constant determined by the formula

$$\nu = -\frac{h}{\lambda}.$$

For example, we calculate the value function, when $f(\tau) = e^{-\tau}$, $g(\tau) = 1/(1 + 0.5e^{-\tau+5})$, $\alpha = 0.5$, $\lambda = 0.1$. The functions $h(\tau)$, $\mu(y)$, $\nu(t)$ (21, 22) are determined in this case as follows

$$h(\tau) = \ln\left(\frac{1}{1 + 0.5e^{-\tau+5}}\right) - \frac{4}{3} e^{-\tau} - 2.099,$$

$$\mu(y) = \frac{4}{3} \ln(y),$$

$$\nu(t) = -\int_t^{+\infty} e^{-\lambda(\tau - t)} h(\tau) d\tau.$$

One can see the plot of the value function on Fig. 1.

Let us note that the constructed value function $\varphi(t, y)$ has properties postulated in theorems and lemmas of the paper.

According to the explicit expressions for the value function $\varphi(t, y)$ (20) one can conclude that in the considered model the optimal result has the decomposition property. Namely, the first term μ depends only on the discount parameter λ, the elasticity of substitution α in the utility function (18), and in the logarithmic way on the initial production y, and does not depend on the specific characters of functions $f(\tau)$ and $g(\tau)$ in the dynamical system (17). On the contrary, the second term ν is determined mainly by dynamics (17) aggregated in the function $h(\tau)$ (22) and does not depend on the initial production y.

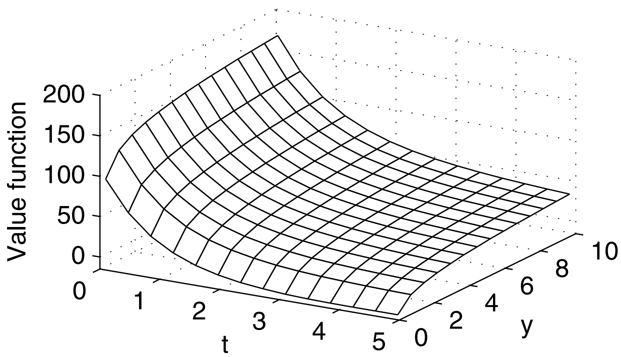

Fig. 1. Value function plot for the case $f(\tau) = e^{-\tau}$, $g(\tau) = 1/(1 + 0.5e^{-\tau+5})$, $\alpha = 0.5$, $\lambda = 0.1$.

5. CONCLUSION

The main result of the article is the necessary and sufficient conditions for the value function. It is shown that these necessary and sufficient conditions are based on the stability properties. It is also demonstrated that the value function is the unique minimax solution of the Hamilton–Jacobi equation. These results can be used to obtain accuracy estimates of grid approximation schemes for solutions of Hamilton–Jacobi equations.

Besides, the article considers important special cases for logarithmic, power and exponential quality functionals that arise in economic growth models (Intriligator, 2002). An example is provided to illustrate the growth asymptotic behavior of the value function.

Finally, let us conclude with the remark that the obtained analytical formulas for the value function design via the Hamilton-Jacobi equations methodology can be effectively used for econometric calculations basing on the real data and for prognostic modeling of economic growth trends.

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